

THE GENERAL LINEAR GROUP AS A COMPLETE INVARIANT FOR C*-ALGEBRAS

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ABSTRACT. In 1955 Dye proved that two von Neumann factors not of type I_{2n} are isomorphic (via a linear or a conjugate linear *-isomorphism) if and only if their unitary groups are isomorphic as abstract groups. We consider an analogue for C*-algebras. We show that the topological general linear group is a classifying invariant for simple, unital AH-algebras of slow dimension growth and of real rank zero, and the abstract general linear group is a classifying invariant for unital Kirchberg algebras in UCT.

1. INTRODUCTION

Since the introduction of the Elliott invariant as a classifying invariant for C*-algebras the classification program for C*-algebras has been rapidly evolving. New invariants were introduced to enrich the program, some more general, and other tailored to specific applications.

For a large class of simple, amenable, unital, separable C*-algebras, Al-Rawashdeh, Booth and the first named author showed in [1] that their unitary group forms a classifying invariant: From an isomorphism of the unitary group of such algebras, they deduced an isomorphism of their Elliott invariant.

In this paper we look at the general linear group (i.e. the group of invertible elements) of a unital C*-algebra as invariant. We extend the result of [1] by replacing the unitary group by the general linear group of a unital C*-algebra. For each unital C*-algebra A we will denote its general linear group by $GL(A)$ and its set of idempotents by $\mathcal{I}(A)$ (see Notation 2.1).

Given two unital C*-algebras A , and B , and a bijection $\varphi: GL(A) \rightarrow GL(B)$ between their general linear groups, the formula

$$1 - 2\theta_\varphi(e) = \varphi(1 - 2e), \quad e \in \mathcal{I}(A),$$

induces a bijection $\theta_\varphi: \mathcal{I}(A) \rightarrow \mathcal{I}(B)$ between the set of idempotents of A and B . This map is not in general an orthoisomorphism of idempotents (i.e. a bijective map which preserves orthogonality of commuting idempotents). It turns out that in many cases θ_φ is essentially an orthoisomorphism. More

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precisely, generalising the notion of oddly decomposability given in [1] (see definition 3.1), we show in Theorem 3.5 that there exist a partitioning of the non-trivial elements $\mathcal{I}(A)$ into two set $\mathcal{I}_o, \mathcal{I}_{\bar{o}}$, such that the map $\tilde{\theta}_\varphi: \mathcal{I}(A) \rightarrow \mathcal{I}(B)$ defined by

$$\tilde{\theta}_\varphi(e) = \begin{cases} \theta_\varphi(e), & \text{if } e \in \mathcal{I}_o \\ 1 - \theta_\varphi(e), & \text{if } e \in \mathcal{I}_{\bar{o}} \\ 1, & \text{if } e = 1 \\ 0, & \text{if } e = 0 \end{cases}$$

is an orthoisomorphism. Using the maps $\tilde{\theta}_\varphi$ and φ between the idempotents and invertibles of A and B we construct appropriate homomorphisms from $K_0(A)$ to $K_0(B)$ and from $K_1(A)$ to $K_1(B)$ and invoke on classification to show A is isomorphic to B . By investing which C^* -algebras are oddly decomposable we prove the following two main results

- (i) Let A and B be simple, unital AH -algebras of slow dimension growth and of real rank zero. Then A and B are isomorphic if and only if their general linear groups are topologically isomorphic.
- (ii) Let A and B be unital Kirchberg algebras in UCT. Then A and B are isomorphic if and only if their general linear groups are isomorphic as abstract groups.

In the case the algebras A and B are simple and finite dimensional we refer to [21] by Schreier and Van der Waerden (see also [12, 15, 22] for related results).

2. PROPERTIES OF THE INDUCED MAP θ_φ

Let A and B be two unital C^* -algebras. If $\varphi: GL(A) \rightarrow GL(B)$ is a group homomorphism between the general linear groups of A and B , then φ defines a map $\theta = \theta_\varphi: \mathcal{I}(A) \rightarrow \mathcal{I}(B)$ by setting

$$1 - 2\theta_\varphi(e) = \varphi(1 - 2e), \quad e \in \mathcal{I}(A).$$

A simple computation shows that $\theta_\varphi(e)$ is an idempotent for each $e \in \mathcal{I}(A)$ making θ_φ well defined. Moreover, if φ is a bijection of invertibles—or simply if φ restricts to a bijection of symmetries ($s^2 = 1$)—it follows that θ_φ is a bijection of idempotents. The following additional properties of the map θ can be easily checked (see [1, 8]).

Proposition 2.1. *Let A and B be unital C^* -algebras, $\varphi: GL(A) \rightarrow GL(B)$ be a group isomorphism and θ be the induced map between idempotents. Then*

- (i) $\theta(ueu^{-1}) = \varphi(u)\theta(e)\varphi(u)^{-1}$,
- (ii) $\theta(0) = 0$,
- (iii) if $e, f \in \mathcal{I}(A)$ commute, then so do $\theta(e)$ and $\theta(f)$ in $\mathcal{I}(B)$,
- (iv) $\theta(e \triangle f) = \theta(e) \triangle \theta(f)$, where \triangle denotes the symmetric difference of commuting idempotents, i.e. $e \triangle f = e + f - 2ef$.

If the center $\mathcal{Z}(B)$ of a unital C^* -algebra B is reduced to the scalars, and $\varphi: GL(A) \rightarrow GL(B)$ is as above, then $\varphi(-1) = -1$. Indeed, note that -1 is a central element which is not 1, but its product with itself equals 1. The same is true for $\varphi(-1)$. As a consequence, we get the following lemma, cf. [1, 8]:

Lemma 2.2. *Let A and B be unital C^* -algebras, whose center $\mathcal{Z}(B) = \mathbb{C}1$. Let $\varphi: GL(A) \rightarrow GL(B)$ be a group isomorphism and $\theta: \mathcal{I}(A) \rightarrow \mathcal{I}(B)$ be as above. Then $\theta(1) = 1$, and for each $e \in \mathcal{I}(A)$, $\theta(1 - e) = 1 - \theta(e)$.*

To simplify notation, let us introduce the following:

Notation 2.1. (i) The quadruple (A, B, φ, θ) will denote a pair of simple unital C^* -algebras A and B , a group isomorphism $\varphi: GL(A) \rightarrow GL(B)$, and the induced bijection $\theta: \mathcal{I}(A) \rightarrow \mathcal{I}(B)$.

(ii) Let A be a unital C^* -algebra. Denote by $\mathcal{I}(A)$ the set of idempotents in A , and by $\widetilde{\mathcal{I}(A)}$ the set $\mathcal{I}(A) \setminus \{0, 1\}$ of *non-trivial* idempotents in A .

(iii) Let A be a unital C^* -algebra. Denote by $GL(A)$ the general linear group of invertible elements in A .

Definition 2.1. Let A be a unital C^* -algebra. We say that two idempotents $e, f \in \mathcal{I}(A)$ are *similar*, denoted $e \sim_s f$, if there exist $u \in GL(A)$ such that $f = ueu^{-1}$.

The following lemma is a generalisation of [8, Lemma 10] to simple, unital C^* -algebras.

Lemma 2.3. *Let (A, B, φ, θ) be as in (2.1). Then for each fixed $e \in \mathcal{I}(A)$*

$$\varphi(\lambda e + 1 - e) \in \mathbb{C}\theta(e) + \mathbb{C}\theta(1 - e), \quad \lambda \in \mathbb{C} \setminus \{0\}.$$

Proof. Fix $\lambda \in \mathbb{C} \setminus \{0\}$ and set $x := \varphi(\lambda e + 1 - e)$. Since every idempotent in B is similar to projection (cf. [2, Proposition 4.6.2]) we can choose $u \in GL(B)$ such that $q = u\theta(e)u^{-1}$ is a projection. For any subset $S \subseteq B$, let S' denote its relative commutant in B , and S'' its (relative) bicommutant.

We show that $uxu^{-1} \in \{q\}'$. Since $q = u\theta(e)u^{-1}$ we just need to show that $x\theta(e) = \theta(e)x$. This follows from $x\varphi(1 - 2e) = \varphi(1 - 2e)x$.

We show $uxu^{-1} \in \{q\}''$. Fix $b \in \{q\}'$. Since q is selfadjoint $\{q\}'$ is a C^* -subalgebra of B and contains unitary elements $\varphi(u_1), \dots, \varphi(u_4)$ that span b , for some $u_1, \dots, u_4 \in GL(A)$. Since $\varphi(u_i) \in \{q\}'$ commutes with $q = u\theta(e)u^{-1}$ we have that $u^{-1}\varphi(u_i)u$ commutes with $\theta(e)$ and with $\varphi(1 - 2e)$. This implies that $\varphi^{-1}(u^{-1})u_i\varphi^{-1}(u)$ commutes with $1 - 2e$, with e and with $\lambda e + 1 - e$. We now have that $u^{-1}\varphi(u_i)u$ commutes with $x = \varphi(\lambda e + 1 - e)$. Therefore $\varphi(u_i)$ commutes with uxu^{-1} , and b commutes with uxu^{-1} . Since b was an arbitrary element in $\{q\}'$ we conclude that uxu^{-1} commutes with every element in $\{q\}'$, i.e. $uxu^{-1} \in \{q\}''$.

Since q is a projection $\{q\}'' \cap \{q\}' = \mathbb{C}q + \mathbb{C}(1 - q)$, using the fact that B is simple so the hereditary C^* -subalgebra qBq is simple and consequently has centre $\mathbb{C}q$ (similarly for $1 - q$). Multiplying uxu^{-1} on the left by u^{-1} and on the right by u we see that $x \in \mathbb{C}\theta(e) + \mathbb{C}\theta(1 - e)$. \square

Lemma 2.4. *Let (A, B, φ, θ) be as in (2.1). Then for each fixed $e \in \widetilde{\mathcal{I}(A)}$ there exist group homomorphisms $a_e, b_e: \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C} \setminus \{0\}$ such that*

$$\varphi(\lambda e + 1 - e) = a_e(\lambda)\theta(e) + b_e(\lambda)\theta(1 - e), \quad \lambda \in \mathbb{C} \setminus \{0\}.$$

Proof. Fix $\lambda \in \mathbb{C} \setminus \{0\}$. Since $e \in \widetilde{\mathcal{I}(A)}$ the elements $\theta(e)$ and $\theta(1 - e)$ are nonzero and linearly independent (cf. Lemma 2.2). Using Lemma 2.3 we therefore have unique coefficients $a, b \in \mathbb{C}$ such that

$$\varphi(\lambda e + 1 - e) = a\theta(e) + b\theta(1 - e).$$

Assuming $b = 0$ we obtain $\varphi(\lambda e + 1 - e)^2 = a^2\theta(e) = a\varphi(\lambda e + 1 - e)$. Hence $a\theta(e) = \varphi(\lambda e + 1 - e) = a1 = a\theta(1)$. This contradicts injectivity of θ or $\varphi(\lambda e + 1 - e)$ being invertible. By symmetry both $a, b \in \mathbb{C} \setminus \{0\}$.

It is easy to see that a_e, b_e are multiplicative and unital using that $\lambda \mapsto \varphi(\lambda e + 1 - e)$ is multiplicative and unital. We conclude both maps are group homomorphisms. \square

Let \mathbb{C}^* denote the group $(\mathbb{C} \setminus \{0\}, \cdot)$. Since the maps a_e, b_e are group homomorphisms of \mathbb{C}^* we will use their (multiplicative) inverses without any further explanation. To each $e \in \widetilde{\mathcal{I}(A)}$, we associate the pair of maps (a_e, b_e) and the group homomorphism $c_e := a_e b_e^{-1}$ of \mathbb{C}^* . Moreover, we denote by \sim_c the equivalence relation on $\widetilde{\mathcal{I}(A)}$, given by:

$$e \sim_c f \quad \text{iff} \quad c_e = c_f$$

Proposition 2.5. *Let (A, B, φ, θ) be as in (2.1). Then for each $e \in \widetilde{\mathcal{I}(A)}$*

- (i) *If $f \in \widetilde{\mathcal{I}(A)}$ is similar to e then $e \sim_c f$.*
- (ii) *$c_e(\lambda)^2 \neq 1$, for every $\lambda \in \mathbb{C} \setminus \{-1, 0, 1\}$.*
- (iii) *$c_e = c_{1-e}$.*

Proof. The proof of [1, Proposition 2.7] generalises (left to reader). \square

Definition 2.2. Two or more idempotents in a C^* -algebra A are *orthogonal* provided that any two of these idempotents commute and their product is equal to zero.

Lemma 2.6. *Let (A, B, φ, θ) be as in (2.1). Suppose that $e, f \in \widetilde{\mathcal{I}(A)}$ are two orthogonal idempotents in A . Then*

$$\begin{aligned} \theta(e + f) &= \theta(e)\theta(1 - f) + \theta(1 - e)\theta(f) \\ \theta(1 - e - f) &= \theta(e)\theta(f) + \theta(1 - e)\theta(1 - f) \end{aligned}$$

Proof. The proof of [1, Lemma 2.3] does not directly generalise, so we include a short proof: Using Proposition 2.1 and Lemma 2.2 we have

$$\theta(e + f) = \theta(e) \triangle \theta(f) = \theta(e) + \theta(f) - 2\theta(e)\theta(f) = \theta(e)\theta(1 - f) + \theta(1 - e)\theta(f).$$

The second equality follows by subtracting both sides of the above equality from 1. \square

Remark 2.1. The proof of Theorem 2.7, Corollary 2.8, and Corollary 2.9 corresponds to the result of [1, Theorem 2.9] and [1, Proposition 2.8], but our proof includes a new characterisation of when $c_e = c_f, \dots, c_e = c_{e+f}^{-1}$ in terms of the equations (2.5)-(2.8). This observation is essential in the subsequent proofs of Corollary 2.8, Corollary 2.9 used to prove Lemma 2.10 and Lemma 2.11.

Theorem 2.7. *Let (A, B, φ, θ) be as in (2.1). Suppose that $e, f \in \widetilde{\mathcal{I}(A)}$ are two orthogonal idempotents in A not adding to one. Then*

$$\begin{aligned} \theta(e)\theta(f) = 0 &\Leftrightarrow c_e = c_f = c_{e+f} \\ \theta(1-e)\theta(1-f) = 0 &\Leftrightarrow c_e = c_f = c_{e+f}^{-1} \\ \theta(1-e)\theta(f) = 0 &\Leftrightarrow c_e = c_f^{-1} = c_{e+f} \\ \theta(e)\theta(1-f) = 0 &\Leftrightarrow c_e = c_f^{-1} = c_{e+f}^{-1} \end{aligned}$$

Proof. Since $\varphi(\lambda e + 1 - e)\varphi(\lambda f + 1 - f) = \varphi(\lambda(e + f) + 1 - (e + f))$ for $\lambda \neq 0$, Lemma 2.4 ensures that

$$(a_e\theta(e) + b_e\theta(1-e))(a_f\theta(f) + b_f\theta(1-f)) = a_{e+f}\theta(e+f) + b_{e+f}\theta(1-e+f).$$

Multiplying this equality by the four idempotents $\theta(e)\theta(f)$, $\theta(1-e)\theta(f)$, $\theta(1-e)\theta(1-f)$, and using Lemma 2.6, we obtain the following four equations

$$(2.1) \quad a_e a_f \theta(e)\theta(f) = b_{e+f} \theta(e)\theta(f)$$

$$(2.2) \quad b_e b_f \theta(1-e)\theta(1-f) = b_{e+f} \theta(1-e)\theta(1-f)$$

$$(2.3) \quad b_e a_f \theta(1-e)\theta(f) = a_{e+f} \theta(1-e)\theta(f)$$

$$(2.4) \quad a_e b_f \theta(e)\theta(1-f) = a_{e+f} \theta(e)\theta(1-f)$$

Consider the following properties

$$(2.5) \quad a_e a_f = b_{e+f}$$

$$(2.6) \quad b_e b_f = b_{e+f}$$

$$(2.7) \quad b_e a_f = a_{e+f}$$

$$(2.8) \quad a_e b_f = a_{e+f}$$

We claim that

$$\begin{aligned} (2.7), (2.8) &\Leftrightarrow c_e = c_f \\ (2.5), (2.6) &\Leftrightarrow c_e = c_f^{-1} \\ (2.6), (2.8) &\Leftrightarrow c_e = c_{e+f} \\ (2.5), (2.7) &\Leftrightarrow c_e = c_{e+f}^{-1} \end{aligned}$$

Going from left to right is straight forward. To go from right to left one simply adds two of the equations (2.1)-(2.4), possibly with coefficients. For

example if $c_e = c_f$ then $a_e b_f = b_e a_f$. By Lemma 2.6 the equality (2.3)+(2.4), where we add each side separately, reduces to

$$a_e b_f \theta(e + f) = b_e a_f \theta(e + f) = a_{e+f} \theta(e + f).$$

Hence (2.7), and (2.8) both hold. The remaining three equivalences are obtained similarly using (2.1) + (2.2), $a_e \cdot (2.2) + b_e \cdot (2.3)$, and $b_e \cdot (2.1) + a_e \cdot (2.3)$. We obtain

$$\begin{aligned} (2.6), (2.7), (2.8) &\Leftrightarrow c_e = c_f = c_{e+f} \\ (2.5), (2.7), (2.8) &\Leftrightarrow c_e = c_f = c_{e+f}^{-1} \\ (2.5), (2.6), (2.8) &\Leftrightarrow c_e = c_f^{-1} = c_{e+f} \\ (2.5), (2.6), (2.7) &\Leftrightarrow c_e = c_f^{-1} = c_{e+f}^{-1} \end{aligned}$$

We now show that $\theta(e)\theta(f) = 0$ if and only if $c_e = c_f = c_{e+f}$. The other three equivalences follow from similar calculations. Suppose first that $\theta(e)\theta(f) = 0$. Suppose for contradiction that $c_e = c_f = c_{e+f}$ does not hold. Then one of (2.6), (2.7), or (2.8) does not hold. Hence $\theta(e)\theta(1-f) = 0$, $\theta(1-e)\theta(f) = 0$, or $\theta(1-e)\theta(1-f) = 0$. Adding this to $\theta(e)\theta(f) = 0$ Lemma 2.6 gives $\theta(e) = 0$, $\theta(f) = 0$, or $\theta(1-e-f) = 0$.

Conversely suppose that the first equation $c_e = c_f = c_{e+f}$ above holds. Since $c_g^2 \neq 1$ for $g = e, f, e+f$, cf. Proposition 2.5(ii), we obtain that all the other three equations are false. Since (2.6), (2.7), and (2.8) hold, (2.5) must fail. We conclude that $\theta(e)\theta(f) = 0$. \square

Corollary 2.8. *Let (A, B, φ, θ) be as in (2.1). Suppose that $e, f \in \widetilde{\mathcal{I}(A)}$ are two orthogonal, \sim_c -equivalent, idempotents in A not adding to one. Then precisely one of $\theta(e)\theta(f)$, $\theta(1-e)\theta(1-f)$ is zero.*

Proof. If both terms are zero then $c_{e+f} = c_{e+f}^{-1}$. If both terms are non-zero then (2.5) and (2.6) are true: If (2.5) fails then $\theta(e)\theta(f) = 0$ by (2.1), and similarly for (2.6). Hence $c_e = c_f^{-1}$ and by \sim_c -equivalence also $c_f = c_f^{-1}$. \square

Corollary 2.9. *Let (A, B, φ, θ) be as in (2.1). Suppose that $e, f \in \widetilde{\mathcal{I}(A)}$ are two orthogonal idempotents in A not adding to one. Then precisely one of $c_e = c_{e+f}$, $c_e = c_{e+f}^{-1}$ is true.*

Proof. Consider the four equations from Theorem 2.7

$$\begin{aligned} c_e = c_f = c_{e+f} & & c_e = c_f = c_{e+f}^{-1} \\ c_e = c_f^{-1} = c_{e+f} & & c_e = c_f^{-1} = c_{e+f}^{-1} \end{aligned}$$

Suppose all four equations above are false. Then all (2.5)-(2.8) are true, because if (2.5) fails then $\theta(e)\theta(f) = 0$ ($\Rightarrow c_e = c_f = c_{e+f}$) and similarly for (2.6)-(2.8). But then all the four equations above are true. Contradiction. Hence some equation is true so $c_e = c_{e+f}$ or $c_e = c_{e+f}^{-1}$. If $c_e = c_{e+f}$ and $c_e = c_{e+f}^{-1}$ we get a contradiction by Proposition 2.5(ii). \square

Definition 2.3. Let (A, B, φ, θ) be as in (2.1). For any subset $S \subseteq \mathcal{I}(A)$ we say that a map

$$\theta: \mathcal{I}(A) \rightarrow \mathcal{I}(B)$$

preserves orthogonality (resp. flips orthogonality) on S if $\theta(e)\theta(f) = 0$ (resp. $\theta(1-e)\theta(1-f) = 0$) for any two orthogonal idempotents e and f in S .

Lemma 2.10. Let (A, B, φ, θ) be as in (2.1). Suppose that $e, f, g \in \widetilde{\mathcal{I}(A)}$ are three orthogonal, \sim_c -equivalent, idempotents in A not adding to one. If θ preserves (resp. flips) orthogonality on a subset of $\{e, f, g\}$ of size two, then θ preserves (resp. flips) orthogonality on all of $\{e, f, g\}$.

Proof. The proof of [1, Lemma 2.14] does not generalise nicely, so we include a short proof:

(1) Suppose that $\theta(e)\theta(f) = 0$, $\theta(e)\theta(g) = 0$. Assume for contradiction that $\theta(1-f)\theta(1-g) \neq 0$. By Lemma 2.6

$$\theta(e)\theta(f+g) = \theta(e)(\theta(f)\theta(1-g) + \theta(1-f)\theta(g)) = 0.$$

Theorem 2.7 implies that $c_e = c_{f+g} = c_{e+f+g}$ and $c_f = c_g = c_{f+g}^{-1}$. Hence $c_{f+g}^2 = 1$, which contradicts Proposition 2.5(ii). We get $\theta(1-f)\theta(1-g) = 0$. By Corollary 2.8 we conclude $\theta(f)\theta(g) = 0$.

(2) Now suppose that $\theta(1-e)\theta(1-f) = 0$, $\theta(1-e)\theta(1-g) = 0$. Assume for contradiction that $\theta(f)\theta(g) \neq 0$. By Lemma 2.6,

$$\theta(1-e)\theta(f+g) = \theta(1-e)(\theta(f)\theta(1-g) + \theta(1-f)\theta(g)) = 0.$$

Theorem 2.7 implies that $c_e = c_{f+g}^{-1} = c_{e+f+g}$ and $c_f = c_g = c_{f+g}$. Hence $c_{f+g}^2 = 1$, which contradicts Proposition 2.5(ii). We get $\theta(f)\theta(g) = 0$. By Corollary 2.8, $\theta(1-f)\theta(1-g) = 0$. (It is evident (1)-(2) are enough.) \square

Lemma 2.11. Let (A, B, φ, θ) be as in (2.1). Suppose that $e, f, g \in \widetilde{\mathcal{I}(A)}$ are three orthogonal, \sim_c -equivalent, idempotents in A not adding to one. Then $e \sim_c f \sim_c g \sim_c e + f + g$.

Proof. The proof of [1, Lemma 2.16] generalises. Notice that use of Lemma 2.6, Corollary 2.8 and Lemma 2.10 provides a few line proof similar to the proof for Lemma 2.10 (left to reader). \square

3. ODDLY DECOMPOSABLE C^* -ALGEBRAS

Let (A, B, φ, θ) be as in (2.1), and let \sim_c be the equivalence relation on $\widetilde{\mathcal{I}(A)}$ introduced in Section 2. We now introduce a sufficient condition on the C^* -algebra A , such that $\widetilde{\mathcal{I}(A)}/\sim_c$ has at most two elements.

Definition 3.1. A unital C^* -algebra A is said to be *oddly decomposable* if for every pair of idempotents $e, f \in \widetilde{\mathcal{I}(A)}$ there is an odd integer $n \geq 3$ and n orthogonal idempotents $g_1, \dots, g_n \in \widetilde{\mathcal{I}(A)}$ adding to f , such that each g_i is similar to some $g'_i \in \widetilde{\mathcal{I}(A)}$ with $g'_i e = e g'_i = g'_i \neq e$.

Remark 3.1. Oddly decomposable C*-algebras were introduced in [1], but with a definition in terms of projections and unitary equivalence rather than idempotents and similarity. In [1] a unital C*-algebra A was called oddly decomposable if for every pair of projections $p, q \in A \setminus \{0, 1\}$ there exist an odd integer $n \geq 3$ and n orthogonal non-zero projections $r_1, \dots, r_n \in A$ adding to q , such that each r_i is unitary equivalent to some projection $r'_i \in A$ with $r'_i < p$. Let us outline why the two definitions coincide:

Fix a pair of idempotents $e, f \in \widetilde{\mathcal{I}(A)}$. Find projections $p, q \in A$ and invertible elements $u, v \in GL(A)$ such that $e = upu^{-1}$, and $f = vqv^{-1}$ (see Lemma 4.1). Assuming odd decomposability in sense of [1] there exist an odd integer $n \geq 3$ and a decomposition of q as a sum $q = \sum_{i=1}^n r_i$ of pairwise nonzero orthogonal projections r_i of A , such that each r_i is unitarily equivalent to some projection $r'_i < p$. Define

$$g_i := vr_i v^{-1}, \quad g'_i := ur'_i u^{-1}, \quad i = 1, \dots, n.$$

It follows that $g_i, g'_i \in \widetilde{\mathcal{I}(A)}$ have the properties needed to make A oddly decomposable in sense of Definition 3.1.

Conversely, fix a pair of projections $p, q \in A \setminus \{0, 1\}$. Assuming odd decomposability in sense of Definition 3.1 there exist an odd integer $n \geq 3$ and n orthogonal idempotents $g_1, \dots, g_n \in \widetilde{\mathcal{I}(A)}$ adding to q , such that each g_i is similar to some $g'_i \in \widetilde{\mathcal{I}(A)}$ with $g'_i p = p g'_i = g'_i \neq p$. We can select $w_1, \dots, w_n \in GL(A)$ such that

$$r_i := w_i g_i w_i^{-1}, \quad i = 1, \dots, n$$

are orthogonal projections adding to q . We can select projections r'_1, \dots, r'_n in A such that each r'_i is similar to g'_i with $r'_i < p$. It follows that each r_i is similar and hence also unitarily equivalent (see Lemma 4.1) to r'_i , making A oddly decomposable in sense of [1].

Notation 3.1. Let A be a unital C*-algebra and $e \in \widetilde{\mathcal{I}(A)}$. We define

$$\mathcal{I}_{c_e} := \{f \in \widetilde{\mathcal{I}(A)} : c_f = c_e\}, \quad \mathcal{I}_{\bar{c}_e} := \{f \in \widetilde{\mathcal{I}(A)} : c_f = c_e^{-1}\}.$$

Lemma 3.1. Let (A, B, φ, θ) be as in (2.1) with A oddly decomposable. Let $e, f \in \widetilde{\mathcal{I}(A)}$ be two idempotents in A . Then there exist idempotents $e', f' \in \widetilde{\mathcal{I}(A)}$ and $u \in GL(A)$ such that

$$e', f' \in \mathcal{I}_{c_f}, \quad e'e = ee' = e' \neq e, \quad f'f = ff' = f' \neq f, \quad e' = uf'u^{-1}.$$

Proof. Despite that the statements of Lemma 3.1 and [1, Corollary 2.17] are different, the proof in [1] can be adopted (left to reader). \square

Lemma 3.2. Let (A, B, φ, θ) be as in (2.1) with A oddly decomposable. Then for each $e \in \widetilde{\mathcal{I}(A)}$

$$\widetilde{\mathcal{I}(A)} = \mathcal{I}_{c_e} \cup \mathcal{I}_{\bar{c}_e}.$$

Proof. The proof of [1, Remark 3.2] does not generalise nicely. We can use Corollary 2.9 and Lemma 3.1 to provide a short proof: Fix any $f \in \widetilde{\mathcal{I}(A)}$. Lemma 3.1 provides an idempotent $e' \in \widetilde{\mathcal{I}(A)}$ such that

$$e' \in \mathcal{I}_{c_f}, \quad e'e = ee' = e' \neq e.$$

Applying Corollary 2.9 to e' and $e - e'$ we get that either $c_{e'} = c_e$ or $c_{e'} = c_e^{-1}$. Hence $c_f = c_e$ or $c_f = c_e^{-1}$. \square

Remark 3.2. Borrowing material from a forthcoming paper [13] let us mention the following result: Let (A, B, φ, θ) be as in (2.1) with φ is continuous. Then for each $e \in \widetilde{\mathcal{I}(A)}$

$$\widetilde{\mathcal{I}(A)} = \mathcal{I}_{c_e} \cup \mathcal{I}_{\bar{c}_e}.$$

Lemma 3.3. *Let (A, B, φ, θ) be as in (2.1) with A oddly decomposable. Let $e, f \in \widetilde{\mathcal{I}(A)}$ be two orthogonal idempotents in A not adding to one. Suppose that θ preserves (resp. flips) orthogonality on $\{e, f\}$. Then θ preserves (resp. flips) orthogonality on all of \mathcal{I}_{c_e} .*

Proof. The proof of [1, Lemma 3.4] generalises. Use of Lemma 2.10 and Lemma 3.1 provides a proof (left to reader). \square

Lemma 3.4. *Let (A, B, φ, θ) be as in (2.1) with A oddly decomposable. Let $e, f \in \widetilde{\mathcal{I}(A)}$ be two idempotents that are not \sim_c -equivalent. If θ preserves (resp. flips) orthogonality on one of the sets $\mathcal{I}_{c_e}, \mathcal{I}_{c_f}$, then θ flips (resp. preserves) orthogonality on the other set.*

Proof. Following [1] notice it suffices to show that

$$(3.1) \quad \theta \text{ can not preserve orthogonality on both } \mathcal{I}_{c_e} \text{ and } \mathcal{I}_{c_f}.$$

$$(3.2) \quad \theta \text{ can not flip orthogonality on both } \mathcal{I}_{c_e} \text{ and } \mathcal{I}_{c_f}.$$

Let us argue why (3.1)-(3.2) suffice: Suppose θ preserves (resp. flips) orthogonality on \mathcal{I}_{c_e} . Using Lemma 3.1 select orthogonal idempotents $g, h \in \mathcal{I}_{c_f}$ not adding to one. By Corollary 2.8 and Lemma 3.3 we obtain that θ either preserves or flips orthogonality on all of $\{g, h\}$, and hence on all of \mathcal{I}_{c_f} . We conclude θ flips (resp. preserves) orthogonality on \mathcal{I}_{c_f} by (3.1)-(3.2).

The proof of [1, Lemma 3.4] generalises in proving (3.1)-(3.2). Use of Lemma 2.11 and Lemma 3.1 provides a proof of (3.1)-(3.2) (left to reader). \square

Theorem 3.5. *Let (A, B, φ, θ) be as in (2.1). If A is oddly decomposable then φ induces an orthoisomorphism between the sets of idempotents $\mathcal{I}(A)$ and $\mathcal{I}(B)$, which preserves similarity of idempotents.*

Proof. We may assume $\widetilde{\mathcal{I}(A)}$ is non-empty. Using Lemma 3.1 select orthogonal, \sim_c -equivalent, idempotents $e, f \in \widetilde{\mathcal{I}(A)}$ not adding to one. Define $o := c_{e+f}$. By Lemma 2.9 either $c_e = o$ or $c_e = o^{-1}$. If $c_e = o^{-1}$ then $c_e = c_f = c_{e+f}^{-1}$ and θ flips orthogonality on \mathcal{I}_{c_e} . If $c_e = o$ then $c_e = c_f = c_{e+f}$

and θ preserves orthogonality on \mathcal{I}_{c_e} . In any case Lemma 3.4 ensures that θ preserves orthogonality on \mathcal{I}_o and flips orthogonality on $\mathcal{I}_{\bar{o}}$. Define

$$\tilde{\theta}(g) = \begin{cases} \theta(g), & \text{if } g \in \mathcal{I}_o \\ 1 - \theta(g), & \text{if } g \in \mathcal{I}_{\bar{o}} \\ 1, & \text{if } g = 1 \\ 0, & \text{if } g = 0 \end{cases}$$

The proof of [1, Theorem 2.21] generalises in proving $\tilde{\theta}$ is orthoisomorphism which preserves similarity of idempotents (left to reader). \square

4. THE CASE OF SIMPLE AH-ALGEBRAS

4.1. From orthoisomorphism to a K_0 -order isomorphism. In this subsection, we prove that an (abstract) isomorphism between the general linear groups of a class of stably finite C^* -algebras of real rank zero (including the simple AH-algebras of slow dimension growth) induces an isomorphism between their ordered K_0 -groups. In particular, we have that if A and B are either two simple unital AF-algebras, or two irrational rotation algebras, then A is $*$ -isomorphic to B if and only if their general linear groups are isomorphic (as abstract groups). Our approach uses ideas of [1], but with proofs that are somehow different, and some clarifications are given.

Notation 4.1. (i) Let \mathcal{F} denote the class of simple, unital, separable C^* -algebras of real rank zero with cancellation (or with stable rank one cf. [2, 6.5.7]) and

$$\mathcal{F}_1 = \{A \in \mathcal{F} : K_0(A) \text{ is noncyclic and weakly unperforated}\}.$$

(ii) Let A be a unital C^* -algebra. Denote by $\mathcal{P}(A)$ the set of projections in A , and by $\widetilde{\mathcal{P}(A)}$ the set $\mathcal{P}(A) \setminus \{0, 1\}$ of *non-trivial* projections in A .

Lemma 4.1. *Let A be a unital C^* -algebra. Every idempotent in A is similar to a projection in A . Every pair of projections in A are similar if, and only if, they are unitary equivalent.*

Proof. Well known. See [2, Proposition 4.6.2, Proposition 4.6.5]. \square

Proposition 4.2. *Each C^* -algebra in \mathcal{F}_1 is oddly decomposable.*

Proof. See Remark 3.1 and [1, Proposition 4.2]. \square

Following [2, 19] an *ordered (abelian) group* G is an abelian group with a distinguished *positive cone*, i.e. a subset $G_+ \subseteq G$ fulfilling that

$$G_+ + G_+ \subseteq G_+, \quad G_+ \cap (-G_+) = \{0\}, \quad G_+ - G_+ = G.$$

The set G_+ induces a translation-invariant partial ordering on G by $x \leq y$ if $y - x \in G_+$.

Essentially¹ as in [9], a *scaled ordered group* G is a ordered group with a distinguished *scale*, *i.e.* a subset $\Gamma = \Gamma(G)$ of G_+ , which is generating, hereditary and directed, *i.e.*

- (i) For each $a \in G_+$, there exist $a_1, \dots, a_r \in \Gamma$ with $a = \sum_{i=1}^r a_i$.
- (ii) If $0 \leq a \leq b \in \Gamma$, then $a \in \Gamma$.
- (iii) Given $a, b \in \Gamma$, there exist $c \in \Gamma$ with $a, b \leq c$.

A scale Γ has a partially defined addition; in fact $a \geq b$ in Γ if, and only if $a = b + c$ for some $c \in \Gamma$. A group homomorphism of scaled ordered groups $\alpha: G \rightarrow G'$ is a *contraction* if $\alpha(\Gamma(G)) \subseteq \Gamma(G')$. If Γ and Γ' are scales of two scaled ordered groups, then (see [9, p. 45]) a map $\alpha: \Gamma \rightarrow \Gamma'$ is a *scale homomorphism* (resp. a *scale isomorphism*) if $a = b + c$ in Γ implies that (resp. is equivalent to) $\alpha(a) = \alpha(b) + \alpha(c)$ in Γ' .

Proposition 4.3 (Effros). *Let G and G' be two scaled ordered groups with Riesz interpolation. Any scale homomorphism $\alpha: \Gamma(G) \rightarrow \Gamma(G')$ extends to a unique contraction $\tilde{\alpha}: G \rightarrow G'$. If α is a scale isomorphism, then $\tilde{\alpha}$ is an isomorphism of the scaled ordered groups G and G' .*

Proof. Notice that the proof of [9, Lemma 7.3, Corollary 7.4] does not use perforation nor countability of the groups involved. \square

Lemma 4.4. *If $A \in \mathcal{F}$, then $K_0(A)$ is a simple scaled ordered group with Riesz interpolation and scale $\Sigma(A) := \{[p]: p \in \mathcal{P}(A)\}$.*

Proof. Since A is stably finite, the group $K_0(A)$ is a simple scaled ordered group with Riesz interpolation by [17, Proposition 3.3.7, Theorem 3.3.18], and $\Sigma(A)$ is hereditary and directed by [2, p. 38].

For sake of completeness we show $\Sigma(A)$ is generating: Fix any x in $K_0(A)_+$. Recall that $x = [p]$ for some projection p in $M_n(A)$ (with $n \in \mathbb{N}$). With $1_n = \sum_{i=1}^n e_{ii}$, where $e_{ii} \in M_n(A)$ is the matrix with 1 at entry (i, i) and zero otherwise, we have that $p \leq 1_n$. Since $M_n(A)$ has real rank zero it follows from [17, Corollary 3.3.17] that there exist projections $p_i \in M_n(A)$ such that $[p_i] \leq [e_{ii}]$ and $\sum_{i=1}^n p_i = p$. Hence $x = \sum_{i=1}^n [p_i]$ and $[p_i] \in \Sigma(A)$, using $\Sigma(A) = \{x \in K_0(A)_+ : x \leq [1]\}$. \square

Theorem 4.5. *Let A and B be two C^* -algebras in \mathcal{F}_1 . If $GL(A)$ and $GL(B)$ are isomorphic (as abstract groups), then $K_0(A)$ and $K_0(B)$ are isomorphic as scaled ordered groups.*

Proof. Let $\tilde{\theta}: \mathcal{I}(A) \rightarrow \mathcal{I}(B)$ be the orthoisomorphism preserving similarity of idempotents given by Theorem 3.5 and Proposition 4.2 with $\tilde{\theta}(1) = 1$. Let

$$\tilde{\theta}_*: \Sigma(A) \rightarrow \Sigma(B)$$

¹Effros presumes the group is unperforated (resp. is a dimension group) in his definition of an ordered group (resp. a scaled dimension group). We have removed these two constraints and changed the terminology accordingly.

be given by $\tilde{\theta}_*([p]) = [p']$, where $p' = u\tilde{\theta}(p)u^{-1}$ for some $u \in GL(B)$ such that p' is a projection in B , cf. Lemma 4.1.

We show $\tilde{\theta}_*$ is well defined: Fix any two projections p, q in $\mathcal{P}(A)$. Assume $[p] = [q]$. By [19, Proposition 3.1.79(ii)] also $[1 - p] = [1 - q]$. Since A has cancellation p and q are unitary equivalent, cf. [19, Definition 7.3.1 and Proposition 2.2.2]. Since $\tilde{\theta}$ preserves similarity $\tilde{\theta}(p)$ and $\tilde{\theta}(q)$ are similar. We conclude that p' and q' are similar and (by Lemma 4.1) unitary equivalent. We obtain that $[p'] = [q']$.

We show $\tilde{\theta}_*$ is a scale homomorphism: Fix $x, y, z \in \Sigma(A)$ with $x + y = z$. Find $p, q \in \mathcal{P}(A)$ such that $x = [p]$ and $y = [q]$. By definition $\tilde{\theta}_*([p]) = [p']$, where $p' = u\tilde{\theta}(p)u^{-1}$ for some $u \in GL(B)$ such that p' is a projection in B . Since p' is a projection, $u\tilde{\theta}(1 - p)u^{-1} = 1 - p'$ is a projection, and $\tilde{\theta}_*([1 - p]) = [1 - p']$. If $z = [1]$ then $[1 - p] = [1] - [p] = z - x = y$. Hence

$$\begin{aligned}\tilde{\theta}_*(z) &= \tilde{\theta}_*([1]) = [1] = [p' + 1 - p'] = [p'] + [1 - p'] \\ &= \tilde{\theta}_*([p]) + \tilde{\theta}_*([1 - p]) = \tilde{\theta}_*(x) + \tilde{\theta}_*(y).\end{aligned}$$

If $[p] + [q] = z < [1]$, then $[p] < [1 - q]$. As A is simple, unital, stably finite, of real rank zero, with cancellation and weakly unperforated $K_0(A)$, [2, Corollary 6.9.2] implies that p is Murray-von Neumann equivalent to a subprojection q_1 of $1 - q$. We obtain that

$$\tilde{\theta}(q_1) + \tilde{\theta}(q) = \tilde{\theta}(q_1 + q).$$

Find $v \in GL(B)$ such that $v\tilde{\theta}(q_1)v^{-1}$ and $v\tilde{\theta}(q)v^{-1}$ are orthogonal projections in B (easy exercise, cf. Remark 3.1). Hence

$$\begin{aligned}\tilde{\theta}_*(z) &= \tilde{\theta}_*([p] + [q]) = \tilde{\theta}_*([q_1] + [q]) = \tilde{\theta}_*([q_1 + q]) \\ &= [v\tilde{\theta}(q_1 + q)v^{-1}] = [v\tilde{\theta}(q_1)v^{-1}] + [v\tilde{\theta}(q)v^{-1}] \\ &= \tilde{\theta}_*([q_1]) + \tilde{\theta}_*([q]) = \tilde{\theta}_*(x) + \tilde{\theta}_*(y)\end{aligned}$$

We show $\tilde{\theta}_*$ is a scale isomorphism: As $\tilde{\theta} : \mathcal{I}(A) \rightarrow \mathcal{I}(B)$ is an orthoisomorphism, its inverse induces a scale homomorphism $(\tilde{\theta}^{-1})_* : \Sigma(B) \rightarrow \Sigma(A)$. For $p \in \mathcal{P}(A)$ we have that

$$\begin{aligned}(\tilde{\theta}^{-1})_*(\tilde{\theta}_*([p])) &= (\tilde{\theta}^{-1})_*[u\tilde{\theta}(p)u^{-1}] \\ &= [v\tilde{\theta}^{-1}(u\tilde{\theta}(p)u^{-1})v^{-1}] \\ &= [vwpw^{-1}v^{-1}] \\ &= [p],\end{aligned}$$

for appropriate $u \in GL(B)$, and $v, w \in GL(A)$, using that $\tilde{\theta}^{-1}$ maps $u\tilde{\theta}(p)u^{-1}$ to an idempotent similar to p . By symmetry both $(\tilde{\theta}^{-1})_* \circ \tilde{\theta}_*$ and $\tilde{\theta}_* \circ (\tilde{\theta}^{-1})_*$ are identity maps. Hence $(\tilde{\theta}^{-1})_* = (\tilde{\theta})_*^{-1}$.

Using Proposition 4.3 and Lemma 4.4 we obtain that $K_0(A)$ and $K_0(B)$ are isomorphic as scaled ordered groups. \square

Lemma 4.6. *Every infinite-dimensional, simple, unital AH-algebras of slow dimension growth and of real rank zero belongs to the class \mathcal{F}_1 .*

Proof. Cancellation: See [3, Theorem 1] and [2, Proposition 6.5.1]. Weakly unperforated: See [23, p. 2]. Noncyclic: See [6, Remark 2.7]. \square

Corollary 4.7. *If A and B are simple, unital AH-algebras of slow dimension growth and of real rank zero, with isomorphic general linear groups (as abstract groups), then $(K_0(A), K_0(A)_+, [1_A])$ and $(K_0(B), K_0(B)_+, [1_B])$ are order isomorphic by a map preserving the distinguished order units.*

Proof. If A is infinite-dimensional then so is B . (By Lemma 4.6 and Proposition 4.2 A is oddly decomposable. We can therefore find arbitrary many orthogonal idempotents (e_i) in $\widetilde{\mathcal{I}(A)}$. The isomorphism of $GL(A)$ and $GL(B)$ induces an orthoisomorphism $\theta: \mathcal{I}(A) \rightarrow \mathcal{I}(B)$, cf. Theorem 3.5. The orthogonal idempotents $\tilde{\theta}(e_i)$ in $\widetilde{\mathcal{I}(B)}$ ensure B is infinite-dimensional.) The desired result follows now from Theorem 4.5. If both A and B are finite dimensional we refer to [12]. \square

Using H. Lin's characterization of TAF-algebras (see [17] or [20, Theorem 3.3.5]) we can also state Corollary 4.7 as follows.

Corollary 4.8. *Let A and B be two simple, unital, nuclear, separable TAF-algebras of real rank zero, in the UCT-class \mathcal{N} with isomorphic general linear groups (as abstract groups), then*

$$(K_0(A), K_0(A)_+, [1_A]) \quad \text{and} \quad (K_0(B), K_0(B)_+, [1_B])$$

are order isomorphic by a map preserving the distinguished order units.

Corollary 4.9. *If A and B are either two simple unital AF-algebras, or two irrational rotation algebras, then A is $*$ -isomorphic to B if and only if their general linear groups are isomorphic (as abstract groups).*

Proof. Both the class of unital simple AF-algebras and the class of irrational rotation algebras are classified by $(K_0, K_{0+}, [1])$, cf. [19, Theorem 7.3.4] and [7, Corollary VI.5.3].

Any unital simple AF-algebras is a nuclear TAF-algebras of real rank zero, in the UCT-class \mathcal{N} , and any irrational rotation algebra is a AH-algebras of slow dimension growth and of real rank zero, cf. [10]. \square

4.2. From a topological general linear group isomorphism to a C^* -isomorphism. For simple AH-algebras of real rank zero, let us recall the classification theorem, provided independently by Gong in [14] and Dadarlat in [5], whose proof uses Elliott-Gong's classification in [11] (see for example [20, Theorem 3.3.1]).

Theorem 4.10 (Dadarlat, Gong, Elliott). *Let A and B be simple, unital, AH-algebras of slow dimension growth and of real rank zero. It follows that A is $*$ -isomorphic to B if and only if*

$$(K_0(A), K_0(A)_+, [1_A]) \cong (K_0(B), K_0(B)_+, [1_B]), \quad K_1(A) \cong K_1(B).$$

Notation 4.2. Let A be a unital C^* -algebra. We equip the general linear group $GL(A)$ with the topology induced by the norm on A . Denote by $GL_0(A)$ the connected component $\{u: u \sim_h 1\}$ of the identity element in $GL(A)$.

Theorem 4.11. *Let A and B be simple, unital AH-algebras of slow dimension growth and of real rank zero. Then A and B are isomorphic if and only if their general linear groups are topologically isomorphic.*

Proof. If A and B are isomorphic then their general linear groups are topologically isomorphic. Conversely let $\varphi: GL(A) \rightarrow GL(B)$ be a topological isomorphism from $GL(A)$ onto $GL(B)$. By continuity of φ we have that $\varphi(GL(A)_0) = GL(B)_0$. It follows that $u + GL(A)_0 \mapsto \varphi(u) + GL(B)_0$ is an isomorphism between $GL(A)/GL(A)_0$ and $GL(B)/GL(B)_0$ (with inverse $v + GL(B)_0 \mapsto \varphi^{-1}(v) + GL(A)_0$). Recall that for a unital C^* -algebra C of stable rank one, $K_1(C)$ is isomorphic to $GL(C)/GL(C)_0$ by [18, Theorem 2.10]. Consequently, we conclude that $K_1(A)$ is isomorphic to $K_1(B)$. \square

5. THE CASE OF KIRCHBERG ALGEBRAS

5.1. From orthoisomorphism to a K_0 -isomorphism. In this subsection, inspired by [1], we show that an isomorphism between the general linear groups of simple, unital, purely infinite C^* -algebras induces an isomorphism between their K_0 -groups.

Theorem 5.1. *Every simple, unital, purely infinite C^* -algebra is oddly decomposable.*

Proof. See Remark 3.1 and [1, Theorem 5.2]. \square

Recall that if A is a purely infinite simple C^* -algebra, then every nonzero projection in A is infinite, and $K_0(A) = \{[p]: p \in \mathcal{P}(A), p \neq 0\}$ (see [19, p. 73-85]). If A , in addition, is unital then 1 is an infinite projection and therefore Murray-von Neumann equivalent to a subprojection $q < 1$. Hence $[q] = [1]$, and $K_0(A) = \{[p]: p \in \widetilde{\mathcal{P}(A)}\}$.

Theorem 5.2. *If A and B are two unital, simple, purely infinite C^* -algebras, whose general linear groups are isomorphic (as abstract groups), then there is an isomorphism from $K_0(A)$ to $K_0(B)$, sending $[1_A]$ to $[1_B]$.*

Proof. Let $\tilde{\theta}: \mathcal{I}(A) \rightarrow \mathcal{I}(B)$ be the orthoisomorphism preserving similarity of idempotents given by Theorem 3.5 and Theorem 5.1 with $\tilde{\theta}(1) = 1$. Let

$$\tilde{\theta}_*: K_0(A) \rightarrow K_0(B)$$

be given by $\tilde{\theta}_*([p]) = [p']$, where $p' = u\tilde{\theta}(p)u^{-1}$ for some $u \in GL(B)$ such that p' is a projection in B , cf. Lemma 4.1.

We show $\tilde{\theta}_*$ is well defined and injective: Fix any two projections p, q in $\widetilde{\mathcal{P}(A)}$. Assume $[p] = [q]$. Since p, q are infinite the assumption is equivalent to p, q being unitary equivalent, cf. [2, Corollary 6.11.9]. By Lemma 4.1 the

assumption is equivalent to p, q being similar. Since $\tilde{\theta}$ and $\tilde{\theta}^{-1}$ preserves similarity the assumption is equivalent to $\tilde{\theta}(p), \tilde{\theta}(q)$ being similar. By definition of p', q' the assumption is equivalent to p', q' being similar, and hence unitary equivalent (cf. Lemma 4.1). Using [2, Corollary 6.11.9] once more we obtain that the assumption $[p] = [q]$ is equivalent to $[p'] = [q']$.

We show $\tilde{\theta}_*$ is unital and surjective: Since $\tilde{\theta}(1) = 1$ is a projection we get that $\tilde{\theta}_*([1]) = [\tilde{\theta}(1)] = [1]$. Fix a projection $p \in \mathcal{P}(B)$. Find an idempotent $e \in \mathcal{I}(A)$ such that $\tilde{\theta}(e) = p$. By Lemma 4.1 there exist a $u \in GL(A)$ such that ueu^{-1} is a projection. Now

$$\tilde{\theta}_*([ueu^{-1}]) = [v\tilde{\theta}(ueu^{-1})v^{-1}],$$

for an appropriate $v \in GL(B)$. Since e is similar to ueu^{-1} then $\tilde{\theta}(e)$ is similar to $v\tilde{\theta}(ueu^{-1})v^{-1}$. By Lemma 4.1 similar projections are unitary equivalent. Hence [2, Corollary 6.11.9] ensures $\tilde{\theta}_*([ueu^{-1}]) = [p]$.

We show $\tilde{\theta}_*$ is a homomorphism: Fix any $p, q \in \mathcal{P}(A)$. Since $1 - q, q$ are (full and properly) infinite we can find projections $r \leq 1 - q$ and $s \leq q$ in A such that p (resp. q) is Murray-von Neumann equivalent to r (resp. s), cf. [19, p. 75]. In particular $[p] = [r]$ and $[q] = [s]$ (cf. [19, p. 40]). Since r and s are orthogonal $[r + s] = [r] + [s]$. Find $v \in GL(B)$ such that $v\tilde{\theta}(r)v^{-1}$ and $v\tilde{\theta}(s)v^{-1}$ are orthogonal projections in B (cf. Remark 3.1). Hence

$$\begin{aligned} \tilde{\theta}_*([p] + [q]) &= \tilde{\theta}_*([r] + [s]) = \tilde{\theta}_*([r + s]) = [v\tilde{\theta}(r + s)v^{-1}] \\ &= [v\tilde{\theta}(r)v^{-1} + v\tilde{\theta}(s)v^{-1}] = [v\tilde{\theta}(r)v^{-1}] + [v\tilde{\theta}(s)v^{-1}] \\ &= \tilde{\theta}_*([r]) + \tilde{\theta}_*([s]) = \tilde{\theta}_*([p]) + \tilde{\theta}_*([q]) \end{aligned}$$

This shows that $\tilde{\theta}_*$ is the desired isomorphism. \square

In [4], J. Cuntz proved that for $2 \leq n < \infty$, $K_0(\mathcal{O}_n) \cong \mathbb{Z}/(n-1)\mathbb{Z}$ and $K_0(\mathcal{O}_\infty) \cong \mathbb{Z}$. Hence, we have:

Corollary 5.3. *Two Cuntz algebras are isomorphic if and only if their general linear groups are isomorphic (as abstract groups).*

5.2. From a general linear group isomorphism to a C*-isomorphism.

Recall that a *Kirchberg algebra* is a purely infinite, simple, nuclear, separable C*-algebra, cf. [20, Definition 4.3.1], and that the following result of Kirchberg and Phillips essentially classifies such algebras.

Theorem 5.4 (Kirchberg, Phillips). *Let A and B be unital Kirchberg algebras in the UCT-class \mathcal{N} . Then A and B are *-isomorphic if, and only if, there exist isomorphisms $\alpha_0: K_0(A) \rightarrow K_0(B)$ and $\alpha_1: K_1(A) \rightarrow K_1(B)$ with $\alpha_0([1_A]) = [1_B]$.*

Notation 5.1. Let A be a unital C*-algebra. As usual, the topology on the unitary group $\mathcal{U}(A)$ is inherited from $GL(A)$. Denote by $\mathcal{U}_0(A)$ the connected component $\{u: u \sim_h 1\}$ of the identity element in $\mathcal{U}(A)$.

Theorem 5.5. *If A and B are two unital, simple, purely infinite C^* -algebras, whose general linear groups are isomorphic (as abstract groups), then the groups $K_1(A)$ and $K_1(B)$ are isomorphic.*

Proof. Let $\varphi: GL(A) \rightarrow GL(B)$ denote the isomorphism of $GL(A)$ and $GL(B)$. Since φ preserves symmetries (i.e. if $s^2 = 1$ in $GL(A)$ then $\varphi(s)^2 = 1$ in $GL(B)$) and symmetries generate the connected component of the identity (cf. [16, Theorem 3.7]) we have that $\varphi(GL_0(A)) = GL_0(B)$. Let

$$\tilde{\varphi}_*: \mathcal{U}(A)/\mathcal{U}_0(A) \rightarrow \mathcal{U}(B)/\mathcal{U}_0(B)$$

be given by $\tilde{\varphi}_*([u]) = [u']$, where $u' = \omega(\varphi(u))$ and ω is the map from [19, Proposition 2.1.8] turning invertible elements into unitaries.

We show $\tilde{\varphi}_*$ is well defined and injective: Fix any two unitaries u, v in $\mathcal{U}(A)$. Assume $[u] = [v]$. Recall that $u \sim_h v$ in $\mathcal{U}(A)$ if, and only if $u \sim_h v$ in $GL(A)$, cf. [19, Proposition 2.1.8]. In particular the assumption is equivalent to $\varphi(u) \sim_h \varphi(v)$ in $GL(B)$ (recalling $\varphi(GL_0(A)) = GL_0(B)$). By [19, Proposition 2.1.8] both $\varphi(u) \sim_h \omega(\varphi(u))$ and $\varphi(v) \sim_h \omega(\varphi(v))$ in $GL(B)$. Hence the assumption is equivalent to $u' \sim_h v'$ in $GL(B)$, and hence also to $[u'] = [v']$, cf. [19, Proposition 2.1.8].

We show $\tilde{\varphi}_*$ is surjective: Fix an unitary $u \in \mathcal{U}(B)$. Find an invertible element $v \in GL(A)$ such that $\varphi(v) = u$. Similarly to a previous argument we have that $\omega(v) \sim_h v = \varphi^{-1}(u)$ in $GL(A)$ and $\varphi(\omega(v)) \sim_h \varphi(v) = u$ in $GL(B)$. Using that $\varphi(\omega(v)) \sim_h \omega(\varphi(\omega(v)))$ we obtain that $[\omega(\varphi(\omega(v)))] = [u]$. Hence $\tilde{\varphi}_*([\omega(v)]) = [u]$.

We show $\tilde{\varphi}_*$ is a homomorphism: Fix any $u, v \in \mathcal{U}(A)$. Using the equivalences $\varphi(u) \sim_h \omega(\varphi(u))$ and $\varphi(v) \sim_h \omega(\varphi(v))$ in $GL(B)$ we obtain that

$$\varphi(u)\varphi(v) \sim_h \varphi(u)\omega(\varphi(v)) \sim_h \omega(\varphi(u))\omega(\varphi(v)) \quad \text{in } GL(B).$$

We also have that $\omega(\varphi(uv)) \sim_h \varphi(uv)$ in $GL(B)$. Combining these relations we have that $[\omega(\varphi(uv))] = [\omega(\varphi(u))\omega(\varphi(v))]$. We conclude that

$$\tilde{\varphi}_*([u])\tilde{\varphi}_*([v]) = [\omega(\varphi(u))\omega(\varphi(v))] = [\omega(\varphi(uv))] = \tilde{\varphi}_*([uv]).$$

This shows that $\tilde{\varphi}_*$ is the desired isomorphism. Recall that for a unital purely infinite simple C^* -algebra C , $K_1(C)$ is isomorphic to $\mathcal{U}(C)/\mathcal{U}_0(C)$ by [4, Theorem 1.9]. Consequently, we conclude that $K_1(A)$ is isomorphic to $K_1(B)$. \square

Thanks to Theorems 5.2 and 5.5, we have the following conclusion:

Corollary 5.6. *Let A and B be two unital Kirchberg algebras in the UCT-class \mathcal{N} . Then A and B are isomorphic if and only if their general linear groups are isomorphic (as abstract groups).*

REFERENCES

1. Ahmed Al-Rawashdeh, Andrew Booth, and Thierry Giordano, *Unitary groups as a complete invariant*, J. Funct. Anal. **262** (2012), no. 11, 4711–4730. MR 2913684

2. Bruce Blackadar, *K-theory for operator algebras*, second ed., Mathematical Sciences Research Institute Publications, vol. 5, Cambridge University Press, Cambridge, 1998. MR 1656031 (99g:46104)
3. Bruce Blackadar, Marius Dădărlat, and Mikael Rørdam, *The real rank of inductive limit C^* -algebras*, Math. Scand. **69** (1991), no. 2, 211–216 (1992). MR 1156427 (93e:46067)
4. Joachim Cuntz, *K-theory for certain C^* -algebras*, Ann. of Math. (2) **113** (1981), no. 1, 181–197. MR 604046 (84c:46058)
5. Marius Dădărlat, *Reduction to dimension three of local spectra of real rank zero C^* -algebras*, J. Reine Angew. Math. **460** (1995), 189–212. MR 1316577 (95m:46116)
6. ———, *Morphisms of simple tracially AF algebras*, Internat. J. Math. **15** (2004), no. 9, 919–957. MR 2106154 (2005i:46061)
7. Kenneth R. Davidson, *C^* -algebras by example*, Fields Institute Monographs, vol. 6, American Mathematical Society, Providence, RI, 1996. MR 1402012 (97i:46095)
8. H. A. Dye, *On the geometry of projections in certain operator algebras*, Ann. of Math. (2) **61** (1955), 73–89. MR 0066568 (16,598a)
9. Edward G. Effros, *Dimensions and C^* -algebras*, CBMS Regional Conference Series in Mathematics, vol. 46, Conference Board of the Mathematical Sciences, Washington, D.C., 1981. MR 623762 (84k:46042)
10. George A. Elliott and David E. Evans, *The structure of the irrational rotation C^* -algebra*, Ann. of Math. (2) **138** (1993), no. 3, 477–501. MR 1247990 (94j:46066)
11. George A. Elliott and Guihua Gong, *On the classification of C^* -algebras of real rank zero. II*, Ann. of Math. (2) **144** (1996), no. 3, 497–610. MR 1426886 (98j:46055)
12. Thierry Giordano and Adam Sierakowski, *On the geometry of idempotents in certain operator algebras*, In Preparation.
13. Thierry Giordano, Aidan Sims, and Adam Sierakowski, *Unitary groups as a complete invariant revisited*, In Preparation.
14. Guihua Gong, *On inductive limits of matrix algebras over higher-dimensional spaces. I, II*, Math. Scand. **80** (1997), no. 1, 41–55, 56–100. MR 1466905 (98j:46061)
15. Jinchuan Hou and Li Huang, *Characterizing isomorphisms in terms of completely preserving invertibility or spectrum*, J. Math. Anal. Appl. **359** (2009), no. 1, 81–87. MR 2542157 (2010g:47073)
16. Michael J. Leen, *Factorization in the invertible group of a C^* -algebra*, Canad. J. Math. **49** (1997), no. 6, 1188–1205. MR 1611648 (2000a:46095)
17. Huaxin Lin, *An introduction to the classification of amenable C^* -algebras*, World Scientific Publishing Co. Inc., River Edge, NJ, 2001. MR 1884366 (2002k:46141)
18. Marc A. Rieffel, *The homotopy groups of the unitary groups of noncommutative tori*, J. Operator Theory **17** (1987), no. 2, 237–254. MR 887221 (88f:22018)
19. M. Rørdam, F. Larsen, and N. Laustsen, *An introduction to K-theory for C^* -algebras*, London Mathematical Society Student Texts, vol. 49, Cambridge University Press, Cambridge, 2000. MR 1783408 (2001g:46001)
20. M. Rørdam and E. Størmer, *Classification of nuclear C^* -algebras. Entropy in operator algebras*, Encyclopaedia of Mathematical Sciences, vol. 126, Springer-Verlag, Berlin, 2002, Operator Algebras and Non-commutative Geometry, 7. MR 1878881 (2002i:46047)
21. O. Schreier and B. L. Van der Waerden, *Die Automorphismen der projektiven Gruppen*, Abh. Math. Sem. Univ. Hamburg **6** (1928), no. 1, 303–322. MR 3069507
22. Peter Šemrl, *Maps on idempotent operators*, Perspectives in operator theory, Banach Center Publ., vol. 75, Polish Acad. Sci., Warsaw, 2007, pp. 289–301. MR 2341352 (2008e:47094)
23. Jesper Villadsen, *The range of the Elliott invariant of the simple AH-algebras with slow dimension growth*, K-Theory **15** (1998), no. 1, 1–12. MR 1643615 (99m:46143)

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